The Poisson Process

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A Poisson Process is a model for a series of discrete events (i.e. jumps) where the average time between events is known, but the exact timing of events is random. The Poisson distribution is used to calculate the probability that k events occur over the time interval $[0, t]$ given that events are independent and occur at a constant rate over a given period of time. The time between each jump is random and exponentially-distributed. In this white paper we will build a model to calculate stock price via a Poisson Process. To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building a model to forecast ABC Company stock price given the following go-forward model assumptions...

Table 1: Go-Forward Model Assumptions

Symbol	Description	Value
S_0	Stock price at time zero ($)	10.00
ϕ	Expected rate of drift (%)	5.00
ω	Jump size mean (%)	2.50
υ	Jump size volatility (%)	6.00
λ	Average number of annual jumps (#)	4.00
t	Time in years (#)	3.00

Our task is to answer the following questions...

Question 1: What is random stock price at the end of year 3 given that there were k = 10 jumps drawn from a Poisson distribution and y = 0.65 drawn from a normal distribution.

Question 2: What is expected unconditional stock price at the end of year 3?

Question 3: Prove your answer to Question 2 above via numerical integration.

Jump Probabilities

We will define the random variable τ to be the arrival time of the next jump, the variable ω to be jump size, and the variable λ to be jump intensity, which is the average number of jumps per unit of time. Assume that we are standing at time zero and want to find the probability that the next jump arrives over time time interval $[t, t + \delta t]$. Given that the number of jumps over the time interval $[0, t]$ is Poisson-distributed and the time between jumps is Exponentially-distributed then from the perspective of time zero the equation for the probability that the next jump arrives at time t is... [1]

$$\mathrm{P}\left[t < \tau \leq t + \delta t\right] = \lambda \,\mathrm{Exp}\left\{ - \lambda \, t \right\} \delta t \tag{1}$$

Using Equation (1) above and Appendix Equation (22) below from the perspective of time zero the equation for the probability that the next jump arrives over the time interval $[0, t]$ is...

$$\mathrm{P}\left[\tau \leq t\right] = \int\limits_0^t \lambda \,\mathrm{Exp}\left\{ - \lambda \, t \right\} \delta t = 1 - \mathrm{Exp}\left\{ - \lambda \, t \right\} \tag{2}$$

Using Equation (2) above from the perspective of time zero the equation for the probability that no jumps arrive over the time interval $[0,t]$ is...

$$\mathrm{P}\bigg[\tau > t\bigg] = 1 - \mathrm{P}\bigg[\tau \le t\bigg] = 1 - \bigg(1 - \mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\bigg) = \mathrm{Exp}\bigg\{-\lambda\,t\bigg\} \tag{3}$$

Using Equations (1) and (3) above from the perspective of time zero the conditional probability that the next jump will arrive at time t given that a jump did not arrive prior to time t is...

$$\mathrm{P}\bigg[t < \tau \le t + \delta t \,\bigg|\, \tau > t\bigg] = \lambda\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\}\,\delta t \bigg/ \mathrm{Exp}\bigg\{-\lambda\,t\bigg\} = \lambda\,\delta t \tag{4}$$

Important: Note that Equation (4) above states that the probability that a jump will occur over any time interval $[t, t+\delta t]$ is always $\lambda\,\delta t$ and is not a function of the length of time interval $[0,t]$, which is the length of time that we have already waited for the next jump to occur. The Exponential distribution is therefore memoryless, which for modeling purposes will prove to be a beneficial mathematical property.

As noted above the number of jumps realized over the time interval $[0,t]$ is Poisson-distributed. The equation for the probability of k jumps over the time interval $[0,t]$ is... [2]

$$\mathrm{Prob}\bigg[k\bigg] = \frac{(\lambda\,t)^k}{k!}\,\mathrm{Exp}\bigg\{-\lambda\,t\bigg\} \tag{5}$$

Conditional Stock Price

We will define the variable J_i to be the i'th jump over the time interval $[0,t]$, the variable ϕ to be the rate of drift, the variable ω to be jump size mean, the variable υ to be jump size volatility, and the variable y_i to be a normally-distributed random variable with mean zero and variance one. The equation for random jump size is..

$$J_i = \bigg(1+\omega\bigg)\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,\upsilon^2 + \upsilon\,y_i\bigg\} \text{ ...where... } y_i \sim N\bigg[0,1\bigg] \tag{6}$$

Note that we can rewrite Equation (6) above as...

$$J_i = \mathrm{Exp}\bigg\{\ln\bigg(1+\omega\bigg)\bigg\}\,\mathrm{Exp}\bigg\{-\frac{1}{2}\,\upsilon^2 + \upsilon\,y_i\bigg\} = \mathrm{Exp}\bigg\{\ln\bigg(1+\omega\bigg) - \frac{1}{2}\,\upsilon^2 + \upsilon\,y_i\bigg\} \tag{7}$$

We will define the variable $S(k)_t$ to be conditional stock price at time t (stock price conditioned on k number of jumps). Using Equations (6) and (7) above the equation for conditional stock price is...

$$\begin{aligned} S(k)_t &= S_0\,\mathrm{Exp}\bigg\{\phi\,t\bigg\}\prod_{i=1}^{k} J_i \\ &= S_0\,\mathrm{Exp}\bigg\{\phi\,t\bigg\}\prod_{i=1}^{k}\mathrm{Exp}\bigg\{\ln\bigg(1+\omega\bigg) - \frac{1}{2}\,\upsilon^2 + \upsilon\,y_i\bigg\} \\ &= S_0\,\mathrm{Exp}\bigg\{\phi\,t\bigg\}\,\mathrm{Exp}\bigg\{k\,\ln\bigg(1+\omega\bigg) - k\,\frac{1}{2}\,\upsilon^2 + \upsilon\sum_{i=1}^{k} y_i\bigg\} \end{aligned} \tag{8}$$

If each normally-distributed random variable y_i in Equation (8) above is independent with mean m and variance v then the following equation holds... [3]

$$\mathbb{E}\bigg[\sum_{i=1}^{k} y_i\bigg] = k\,m + \sqrt{k\,v}\,y \text{ ...where... } y \sim N\bigg[m, v\bigg] \tag{9}$$

Per Equation (6) above the mean and variance of each random variable y_i is zero and one, respectively. We can therefore rewrite Equation (9) above as...

$$\mathbb{E}\bigg[\sum_{i=1}^{k} y_i\bigg] = k \times 0 + \sqrt{k \times 1} \times y = \sqrt{k}\,y \text{ ...where... } y \sim N\bigg[0,1\bigg] \text{ ...because... } m = 0 \text{ ...and... } v = 1 \tag{10}$$

Using Equations (9) and (10) above we can rewrite conditional stock price Equation (8) above as...

$$S(k)_t = S_0 \,\mathrm{Exp}\left\{\phi\, t + k\, \ln(1+\omega) - k\,\frac{1}{2}\,\upsilon^2 + \upsilon\,\sqrt{k}\,y\right\} \text{ ...where... } y \sim N\Big[0,1\Big] \tag{11}$$

Using conditional stock price Equation (11) above and noting that the number of jumps k is given (i.e. not random) then the equation for expected conditional stock price at time t is...

$$\mathbb{E}\Big[S(k)_t\Big] = \mathbb{E}\Big[S_0\,\mathrm{Exp}\left\{\phi\,t + k\,\ln(1+\omega) - k\,\frac{1}{2}\,\upsilon^2 + \upsilon\,\sqrt{k}\,y\right\}\Big] = S_0\,\mathrm{Exp}\left\{\phi\,t + k\,\ln(1+\omega)\right\}\mathbb{E}\Big[\mathrm{Exp}\left\{-k\,\frac{1}{2}\,\upsilon^2 + \upsilon\,\sqrt{k}\,y\right\}\Big] \tag{12}$$

We will define the function $f(k)$ to be the following normally-distributed random variable...

$$f(k) = -k\,\frac{1}{2}\,\upsilon^2 + \upsilon\,\sqrt{k}\,y \text{ ...where... } f(k) \sim N\Big[-k\,\frac{1}{2}\,\upsilon^2, k\,\upsilon^2\Big] \text{ ...when... } y \sim N\Big[0,1\Big] \tag{13}$$

Given that the function $f(k)$ in Equation (13) above is normally-distributed then the exponential of $f(k)$ is lognormally-distributed. The equation for the expected value of the exponential of $f(k)$ is...

$$\mathbb{E}\Big[\mathrm{Exp}\left\{f(k)\right\}\Big] = \mathrm{Exp}\left\{\text{mean} + \frac{1}{2}\,\text{variance}\right\} = \mathrm{Exp}\left\{-k\,\frac{1}{2}\,\upsilon^2 + \frac{1}{2}\,k\,\upsilon^2\right\} = \mathrm{Exp}\left\{0\right\} = 1 \tag{14}$$

Using Equations (13) and (14) above we can rewrite Equation (12) above as...

$$\mathbb{E}\Big[S(k)_t\Big] = S_0\,\mathrm{Exp}\left\{\phi\,t + k\,\ln(1+\omega)\right\} = S_0\,\mathrm{Exp}\left\{\phi\,t\right\}\Big(1+\omega\Big)^k \tag{15}$$

Unconditional Stock Price

Using Equation (5) above the equation for expected unconditional stock price is...

$$\mathbb{E}\Big[S_t\Big] = \sum_{k=0}^{\infty}\frac{(\lambda\,t)^k}{k!}\,\mathrm{Exp}\left\{-\lambda\,t\right\}\mathbb{E}\Big[S(k)_t\Big] \tag{16}$$

Using Equation (15) above we can rewrite Equation (16) above is...

$$\begin{aligned}\mathbb{E}\Big[S_t\Big] &= \sum_{k=0}^{\infty}\frac{(\lambda\,t)^k}{k!}\,\mathrm{Exp}\left\{-\lambda\,t\right\} S_0\,\mathrm{Exp}\left\{\phi\,t\right\}\Big(1+\omega\Big)^k \\ &= S_0\,\mathrm{Exp}\left\{\phi\,t\right\}\sum_{k=0}^{\infty}\frac{(\lambda\,t)^k}{k!}\,\mathrm{Exp}\left\{-\lambda\,t\right\}\Big(1+\omega\Big)^k \\ &= S_0\,\mathrm{Exp}\left\{\phi\,t\right\}\sum_{k=0}^{\infty}\frac{((1+\omega)\,\lambda\,t)^k}{k!}\,\mathrm{Exp}\left\{-\lambda\,t\right\}\end{aligned} \tag{17}$$

Note that the solution to the following series is...

$$\text{given that... } \sum_{k=0}^{\infty}\frac{z^k}{k!} = \mathrm{Exp}\left\{z\right\} \text{ ...then... } \sum_{k=0}^{\infty}\frac{((1+\omega)\,\lambda\,t)^k}{k!} = \mathrm{Exp}\left\{(1+\omega)\,\lambda\,t\right\} \tag{18}$$

Using Equation (18) above the solution to Equation (17) above is...

$$\mathbb{E}\Big[S_t\Big] = S_0\,\mathrm{Exp}\left\{\phi\,t\right\}\mathrm{Exp}\left\{(1+\omega)\,\lambda\,t\right\}\mathrm{Exp}\left\{-\lambda\,t\right\} = S_0\,\mathrm{Exp}\left\{\phi\,t + \omega\,\lambda\,t\right\} \tag{19}$$

The Answers To Our Hypothetical Problem

Question 1: What is random stock price at the end of year 3 given that there were k = 10 jumps drawn from a Poisson distribution and y = 0.65 drawn from a normal distribution.

Using Equation (11) above and the data in Table 1 above the answer to the question is...

$$S(10)_3 = 10.00 \times \text{Exp}\left\{0.05 \times 3 + 10 \times \ln(1 + 0.025) - 10 \times \frac{1}{2} \times 0.06^2 + 0.06 \times \sqrt{10} \times 0.65\right\} = 16.52 \tag{20}$$

Question 2: What is expected unconditional stock price at the end of year 3?

Using Equation (19) above and the data in Table 1 above the answer to the question is...

$$\mathbb{E}\left[S_3\right] = 10.00 \times \text{Exp}\left\{(0.05 + 0.025 \times 4) \times 3\right\} = 15.68 \tag{21}$$

Question 3: Prove your answer to Question 2 above via numerical integration.

Using Poisson probability Equation (5) above and expected conditional stock price Equation (15) above the answer to Question 2 above via numerical integration is...

Jumps	Probability	Stock Price	Wt Price
0	0.00001	11.62	0.00007
1	0.00007	11.91	0.00088
2	0.00044	12.21	0.00540
3	0.00177	12.51	0.02214
4	0.00531	12.82	0.06808
5	0.01274	13.15	0.16748
6	0.02548	13.47	0.34333
7	0.04368	13.81	0.60328
8	0.06552	14.16	0.92754
9	0.08736	14.51	1.26763
10	0.10484	14.87	1.55919
11	0.11437	15.24	1.74346
12	0.11437	15.63	1.78704
13	0.10557	16.02	1.69082
14	0.09049	16.42	1.48550
15	0.07239	16.83	1.21811
16	0.05429	17.25	0.93642
17	0.03832	17.68	0.67753
18	0.02555	18.12	0.46298
19	0.01614	18.57	0.29972
20	0.00968	19.04	0.18433
21	0.00553	19.51	0.10796
22	0.00302	20.00	0.06036
23	0.00157	20.50	0.03228
24	0.00079	21.01	0.01654
25	0.00038	21.54	0.00814
26	0.00017	22.08	0.00385
27	0.00008	22.63	0.00175
28	0.00003	23.20	0.00077
29	0.00001	23.78	0.00033
30	0.00001	24.37	0.00013
Total	1.00000	–	15.68304

References

[1] Gary Schurman, *Modeling Exponential Arrival Times,* September, 2015.

[2] Gary Schurman, *The Poisson Distribution,* June, 2012.

[3] Gary Schurman, *The Mean and Variance of the Sum of Normally-Distributed Independent Random Variables,* June, 2015.

Appendix

A. The solution to the following integral is...

$$\int_0^t \lambda \,\mathrm{Exp}\left\{-\lambda\, t\right\}\delta t = -\mathrm{Exp}\left\{-\lambda\, t\right\}\Bigg[_0^t = -\left(\mathrm{Exp}\left\{-\lambda\, t\right\} - 1\right) = 1 - \mathrm{Exp}\left\{-\lambda\, t\right\} \tag{22}$$